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The effect of rigid rotation on the finite-amplitude stability of pipe flow at high Reynolds number

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A theoretical study is made of the stability of pipe flow with superimposed rigid rotation to finite-amplitude disturbances at high Reynolds number. The non-axisymmetric mode that requires the least amount of rotation for linear instability is considered. An amplitude expansion is developed close to the corresponding neutral stability curve; the appropriate Landau constant is calculated. It is demonstrated that the flow exhibits nonlinear subcritical instability, the nonlinear effects being particularly strong owing to the large magnitude of the Landau constant. These findings support the view that a small amount of extraneous rotation could play a significant role in the transition to turbulence of pipe flow.

1. Introduction

The instability of laminar pipe flow and its transition to turbulence, first observed by Reynolds (1883), marked the starting point of extensive theoretical and experimental efforts to understand the mechanisms by which laminar shear flows undergo transition to turbulence. A century of scientific investigation has revealed much about this complex phenomenon in a number of flows. Despite many attempts, however, the instability of fully developed Poiseuille flow in a circular pipe has defied an entirely satisfactory theoretical explanation.

The main obstacle to gaining an understanding of pipe-flow instability is the well-accepted fact that linear-stability theory cannot provide a finite critical Reynolds number: all infinitesimal disturbances are damped at all Reynolds numbers (Salwen, Cotton & Grosch 1980). On the other hand, experimentally, pipe flow can indeed be kept laminar up to quite high Reynolds numbers (about 13000 according to Leite 1959) if only axisymmetric perturbations are allowed, but undergoes transition at a Reynolds number of about 2000 when non-axisymmetric disturbances are introduced (Bhat 1966). The inadequacy of linear stability theory to account for the observed instability points to the relevance of finite-amplitude disturbances. However, various attempts to pinpoint the destabilizing nature of finite-amplitude effects for linearly damped axisymmetric disturbances (Davey & Nguyen 1971; Itoh 1977) through amplitude expansions of the Stuart–Watson type have given inconsistent results (Davey 1978), since there is no neutral-stability curve about which to perturb (Herbert 1983). In fact, direct numerical computations by Patera & Orszag (1981) indicate that all finite-amplitude axisymmetric disturbances in pipe flow are damped; more recent numerical experiments (Orszag & Patera 1983) suggest the possibility of a secondary instability akin to the one found in plane Poiseuille flow: infinitesimal non-axisymmetric perturbations superposed on decaying finite-amplitude axisymmetric states may amplify at high-enough Reynolds numbers (higher than about 4000), but the origin of the assumed axisymmetric states still remains unclear. On the other hand, based on an asymptotic analysis, Smith & Bodonyi (1982) infer that finite-amplitude non-axisymmetric (azimuthal wavenumber equal to one) neutral disturbances exist in pipe flow at high Reynolds number, so that the nonlinear effects dominate the viscous effects at the critical layer. However, as their work deals solely with neutral disturbances, it is not clear whether these neutral disturbances represent isolated finite-amplitude states or stability boundaries.

The motivation for this work comes from the findings of Mackrodt (1976) and Cotton & Salwen (1981) concerning the stability of pipe flow with superimposed rigid rotation; they point out that the addition of rigid rotation makes pipe flow linearly unstable to non-axisymmetric disturbances. Mackrodt suggests that, at high Reynolds number (when instability in pipe flow is observed experimentally), the small amount of rotation required to destabilize pipe flow could have escaped notice in experiment. In such a case, the question arises as to whether finite-amplitude effects tend to further destabilize the flow, so that nonlinear subcritical instability occrus as in plane Poiseuille flow, or the initial exponential growth predicted by the linear theory is eventually saturated. In the present paper, this question is addressed by investigating the nonlinear evolution of the non-axisymmetric mode that requires the least amount of rotation for linear instability. A nonlinear amplitude expansion is developed close to the neutral-stability curve predicted by the linear theory, so that no inconsistencies of the type previously encountered in non-rotating pipe flow (Davey 1978) are present. It is found that the nonlinear effects are particularly strong: the calculated Landau constants are surprisingly large in magnitude owing to the large mean-flow distortion induced by the fluctuating Reynolds stresses. Thus, nonlinear subcritical instability occurs: the flow is nonlinearly unstable to non-axisymmetric perturbations even when the rotation is smaller than that required for linear instability. These findings support the suggestion of Mackrodt (1976) that a very small amount of extraneous rotation could play an important role in the transition to turbulence of pipe flow.

2. Formulation and linear stability analysis

Consider an infinitely long circular pipe of radius r_0, $0 < r < r_0$, $-\infty < z < \infty$. The basic laminar flow consists of a parabolic velocity profile in the axial direction and a rigid-body rotation in the azimuthal direction; in cylindrical polar coordinates the basic-flow velocity components are $(\overline{u}, \overline{v}, \overline{w}) = (0, \omega r, W_0(1-r^2/r_0^2))$, where W_0 is the centreline axial velocity and ω is the constant rotation rate. The corresponding pressure follows from the Navier–Stokes equations:

$$\overline{p} = \tfrac{1}{2}\rho\omega^2 r^2 - \frac{4W_0 \nu\rho}{r_0^2} z, \tag{1}$$

where ρ is the fluid density and ν is the kinematic viscosity.

The stability of rotating-pipe flow is examined by investigating the evolution of perturbations to the basic steady flow:

$$\left.\begin{aligned} \mathscr{U} &= \overline{u}+u(r,\theta,z,t), \quad \mathscr{V} = \overline{v}+v(r,\theta,z,t), \\ \mathscr{W} &= \overline{w}+w(r,\theta,z,t), \quad \mathscr{P} = \overline{p}+p(r,\theta,z,t), \end{aligned}\right\} \tag{2}$$

$\mathscr{U}, \mathscr{V}, \mathscr{W}$ being the radial, azimuthal and axial velocity components respectively, and $\mathscr{P}$ the pressure. It proves convenient to use dimensionless (primed) variables:

$$\left.\begin{aligned} r = r_0 r', \quad z = lz', \quad u = \mu\epsilon W_0 u', \quad v = \mu\epsilon W_0 v', \\ w = \epsilon W_0 w', \quad t = \frac{l}{W_0} t', \quad p = \mu^2 \epsilon\rho W_0^2 p', \end{aligned}\right\} \tag{3}$$

where l is the typical lengthscale of variation of the perturbations in the axial direction, and $\mu = r_0/l$; the dimensionless parameter ϵ quantifies the ratio of the typical axial perturbation velocity to the centreline basic-flow velocity W_0, and is a measure of nonlinearity.

The governing dimensionless equations for the perturbation quantities are obtained by substituting (2) and (3) into the continuity and the Navier–Stokes equations. Our interest centres on the nonlinear behaviour of those perturbations that require the least amount of rotation to become linearly unstable. The linear stability analysis of Mackrodt (1976) shows that such perturbations, which are non-axisymmetric, vary slowly in the axial direction ($\mu \to 0$) and are found as the Reynolds number $R = W_0 r_0/\nu$ becomes large ($R \to \infty$), so that the modified Reynolds number $\tilde{R} = \mu R$ remains finite. In addition, the relative magnitudes of the radial and azimuthal perturbation velocity components are $O(\mu)$ compared with the axial perturbation velocity component; this relation is brought out by the scalings chosen in (3). Accordingly, in the limit $\mu \to 0$ and $\tilde{R}$ finite, the perturbation equations, in terms of the transformed velocity components $(\tilde{u}, \tilde{v}, \tilde{w}) = r'(u', v', w')$ of Mackrodt (when the primes are dropped) read

$$\frac{\partial \tilde{u}}{\partial r} + \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta} + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{4}$$

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\Omega}{\tilde{R}}\frac{\partial \tilde{u}}{\partial \theta} + (1-r^2)\frac{\partial \tilde{u}}{\partial z} - 2\frac{\Omega}{\tilde{R}}\tilde{v} + r\frac{\partial p}{\partial r} - \frac{1}{\tilde{R}}\left(\frac{\partial^2 \tilde{u}}{\partial r^2} - \frac{1}{r}\frac{\partial \tilde{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial \tilde{v}}{\partial \theta}\right) + \epsilon N_1 = 0, \tag{5}$$

$$\frac{\partial \tilde{v}}{\partial t} + \frac{\Omega}{\tilde{R}}\frac{\partial \tilde{v}}{\partial \theta} + (1-r^2)\frac{\partial \tilde{v}}{\partial z} + 2\frac{\Omega}{\tilde{R}}\tilde{u} + \frac{\partial p}{\partial \theta} - \frac{1}{\tilde{R}}\left(\frac{\partial^2 \tilde{v}}{\partial r^2} - \frac{1}{r}\frac{\partial \tilde{v}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{v}}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial \tilde{u}}{\partial \theta}\right) + \epsilon N_2 = 0, \tag{6}$$

$$\frac{\partial \tilde{w}}{\partial t} + \frac{\Omega}{\tilde{R}}\frac{\partial \tilde{w}}{\partial \theta} + (1-r^2)\frac{\partial \tilde{w}}{\partial z} - 2r\tilde{u} - \frac{1}{\tilde{R}}\left(\frac{\partial^2 \tilde{w}}{\partial r^2} - \frac{1}{r}\frac{\partial \tilde{w}}{\partial r} + \frac{\tilde{w}}{r^2} + \frac{1}{r^2}\frac{\partial^2 \tilde{w}}{\partial \theta^2}\right) + \epsilon N_3 = 0, \tag{7}$$

where

$$\Omega = \frac{\omega r_0}{\mu W_0}\tilde{R} \tag{8}$$

is the azimuthal Reynolds number, and N_1, N_2, N_3 are the nonlinear terms

$$N_1 = \frac{1}{r}\left[\tilde{u}\left(\frac{\partial \tilde{u}}{\partial r} - \frac{\tilde{u}}{r}\right) + \frac{\tilde{v}}{r}\frac{\partial \tilde{u}}{\partial \theta} + \tilde{w}\frac{\partial \tilde{u}}{\partial z} - \frac{\tilde{v}^2}{r}\right], \tag{9}$$

$$N_2 = \frac{1}{r}\left[\tilde{u}\frac{\partial \tilde{v}}{\partial r} + \frac{\tilde{v}}{r}\frac{\partial \tilde{v}}{\partial \theta} + \tilde{w}\frac{\partial \tilde{v}}{\partial z}\right], \tag{10}$$

$$N_3 = \frac{1}{r}\left[\tilde{u}\left(\frac{\partial \tilde{w}}{\partial r} - \frac{\tilde{w}}{r}\right) + \frac{\tilde{v}}{r}\frac{\partial \tilde{w}}{\partial \theta} + \tilde{w}\frac{\partial \tilde{w}}{\partial z}\right]. \tag{11}$$

In addition, the perturbation quantities should be regular at the centre of the pipe, and the velocity components should satisfy the no-slip condition at the wall:

$$\tilde{u} = \tilde{v} = \tilde{w} = 0 \quad (r = 1). \tag{12}$$

The linearized perturbation equations follow from (4)–(7) by setting $\epsilon = 0$; the linear stability analysis proceeds in the usual way by considering normal modes:

$$(\tilde{u}, \tilde{v}, \tilde{w}) = (U(r), V(r), W(r))\, \mathrm{e}^{\mathrm{i}\phi}, \quad p = P(r)\, \mathrm{e}^{\mathrm{i}\phi}, \tag{13}$$

where

$$\phi = z + \theta - ct. \tag{14}$$

Note that, without any loss of generality, the axial wavenumber has been normalized to unity; this choice fixes the lengthscale l. Furthermore, the azimuthal wavenumber is also taken to be equal to unity, for, as shown by Mackrodt (1976), those modes are the most unstable and are of primary interest in the subsequent analysis.

Substitution of (13) into (4)–(7) with $\epsilon = 0$ leads to a system of ordinary differential equations for the modal amplitudes U, V, W, P:

$$\frac{\mathrm{d}}{\mathrm{d}r} \boldsymbol{X} = \boldsymbol{K} \boldsymbol{X}, \tag{15}$$

where $\boldsymbol{X}$ is a column vector with entries U, V, W, P, F, G; F and G being the auxiliary variables $F = \mathrm{d}V/\mathrm{d}r$, $G = \mathrm{d}W/\mathrm{d}r$, and $\boldsymbol{K}$ the 6×6 matrix with non-zero elements

$$\left.\begin{aligned} & k_{12} = -\frac{\mathrm{i}}{r}, \quad k_{13} = -\mathrm{i}, \quad k_{25} = 1, \quad k_{36} = 1, \\ & k_{41} = -\frac{1}{\tilde{R}}\left(\frac{1}{r^3} + \frac{\alpha}{r} + \beta r\right), \quad k_{42} = \frac{2\Omega}{\tilde{R} r}, \quad k_{43} = \frac{\mathrm{i}}{\tilde{R} r^2}, \\ & k_{45} = \frac{-\mathrm{i}}{\tilde{R} r^2}, \quad k_{46} = \frac{-\mathrm{i}}{\tilde{R} r}, \quad k_{51} = -2\left(\frac{\mathrm{i}}{r^2} - \Omega\right), \\ & k_{52} = \frac{1}{r^2} + \alpha + \beta r^2, \quad k_{54} = \mathrm{i}\tilde{R}, \quad k_{55} = \frac{1}{r}, \\ & k_{61} = -2\tilde{R} r, \quad k_{63} = \alpha + \beta r^2, \quad k_{66} = \frac{1}{r}, \end{aligned}\right\} \tag{16}$$

where

$$\alpha = \mathrm{i}\tilde{R}(1-c) + \mathrm{i}\Omega, \quad \beta = -\mathrm{i}\tilde{R}. \tag{17}$$

The system of equations (15), together with the regularity conditions at the centre of the pipe and the no-slip condition at the wall,

$$U = V = W = P = 0 \quad (r = 0), \tag{18}$$

$$U = V = W = 0 \quad (r = 1), \tag{19}$$

form an eigenvalue problem for $c = c_{\mathrm{r}} + \mathrm{i}c_{\mathrm{i}}$, which has been analysed in detail by Mackrodt (1976) and Cotton & Salwen (1981). They find that linear instability ($c_{\mathrm{i}} > 0$) requires $\Omega < 0$ (i.e. the swirl of the unstable perturbations is in the direction opposite to the rigid rotation Ω). The neutral-stability curve in terms of Ω and $\tilde{R}$ is reproduced in figure 1; the minimum rotation (in absolute value) for neutral stability is $\Omega = -29.96$ at $\tilde{R} = 106.6$.

It is clear from the definition of Ω in (8) that, on the linear neutral stability curve, the ratio of the azimuthal basic-flow velocity component at the wall, ωr_0, to the axial centreline velocity W_0 is $O(\mu)$. Thus, as first pointed out by Mackrodt (1976), the linear stability theory suggests that, at high Reynolds number, a relatively small azimuthal velocity component is capable of destabilizing pipe flow. However, the behaviour of a linearly unstable disturbance in the finite-amplitude regime cannot be inferred without assessing the importance of the neglected nonlinear terms. In §3 a consistent finite-amplitude expansion is developed close to the linear neutral stability curve.

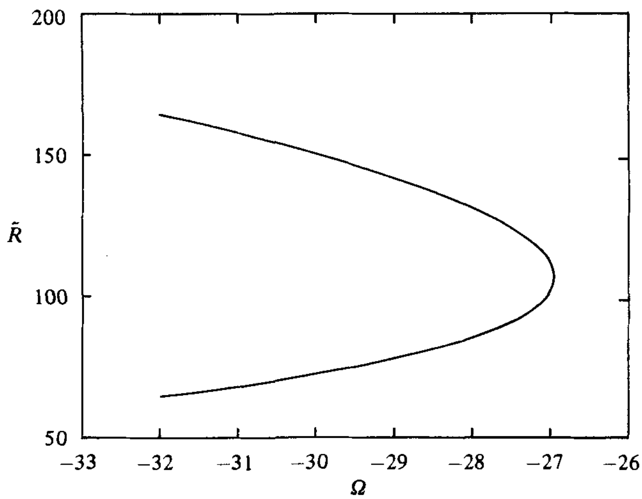

FIGURE 1. The linear neutral-stability curve.

3. Finite-amplitude stability analysis

In the finite-amplitude regime, $0 < \epsilon \ll 1$, a linearly neutral disturbance can be formally expanded in a Stuart–Watson-type expansion (Stuart 1960; Watson 1960):

$$\boldsymbol{x} = (A(T)\boldsymbol{X}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{c.c.}) + \epsilon(A^2\boldsymbol{X}^{(2)}\mathrm{e}^{2\mathrm{i}\phi} + \mathrm{c.c.}) + \epsilon|A|^2\boldsymbol{X}^{(0)} + O(\epsilon^2), \tag{20}$$

where $\boldsymbol{x}$ is the column vector with entries $\tilde{u}, \tilde{v}, \tilde{w}, p, \partial\tilde{v}/\partial r, \partial\tilde{w}/\partial r$, and c.c. denotes the complex conjugate. The leading-order term in (20) is the result of the linear theory outlined above, with the modal vector $\boldsymbol{X}$ normalized so that $W = 1$, where $|W|$ attains its maximum value in $0 < r < 1$. (Figures 2a, b, c show $U(r), V(r), W(r)$ for $\Omega = -26.96$, $\tilde{R} = 106.6$.) The amplitude of the primary harmonic $A(T)$ is a function of the slow timescale $T = \epsilon^2 t$ owing to finite-amplitude effects. The $O(\epsilon)$ terms in (20) represent the second harmonic (proportional to A^2), generated by the self-interaction of the primary harmonic, and the induced mean-flow distortion (proportional to $|A|^2$).

The aim of the following analysis is to investigate the behaviour of the amplitude A with T. To this end, the second harmonic and the mean-flow distortion need to be specified first.

3.1. *The second harmonic*

Substitution of the proposed expansion (20) into the governing equations (4)–(7) and the boundary conditions shows that the modal vector $\boldsymbol{X}^{(2)}$ of the $O(\epsilon)$ second-harmonic response satisfies the inhomogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}r}\boldsymbol{X}^{(2)} - \boldsymbol{K}^{(2)}\boldsymbol{X}^{(2)} = \boldsymbol{Q}^{(2)}, \tag{21}$$

$$U^{(2)} = V^{(2)} = W^{(2)} = P^{(2)} = 0 \quad (r = 0), \tag{22}$$

$$U^{(2)} = V^{(2)} = W^{(2)} = 0 \quad (r = 1); \tag{23}$$

the forcing term $\boldsymbol{Q}^{(2)}$ is a column vector with non-zero entries

$$Q_4^{(2)} = \frac{1}{r^3}(U^2 + V^2), \tag{24}$$

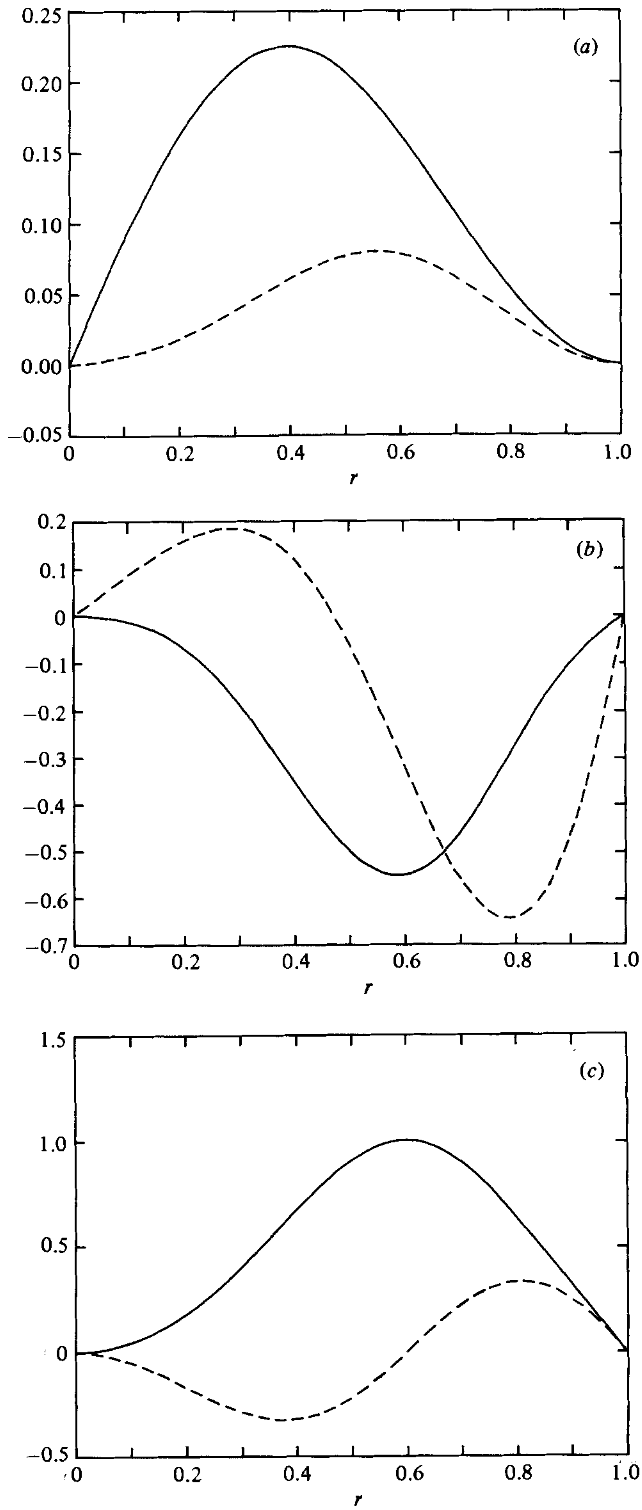

FIGURE 2. The modal amplitudes of the primary harmonic for $\Omega = -26.96$, $\tilde{R} = 106.6$ (——, real part; - - -, imaginary part): (a) radial velocity component; (b) azimuthal velocity component; (c) axial velocity component.

$$Q_5^{(2)} = \frac{\tilde{R}}{r}\left(UF + \mathrm{i}\frac{V^2}{r} + \mathrm{i}VW\right), \tag{25}$$

$$Q_6^{(2)} = \frac{\tilde{R}}{r}\left[U\left(G - \frac{W}{r}\right) + \mathrm{i}\frac{VW}{r} + \mathrm{i}W^2\right], \tag{26}$$

and $\boldsymbol{K}^{(2)}$ is a 6×6 matrix with non-zero elements

$$\left.\begin{aligned}
&k_{12}^{(2)} = -\frac{2\mathrm{i}}{r}, \quad k_{13}^{(2)} = -2\mathrm{i}, \quad k_{25}^{(2)} = 1, \quad k_{36}^{(2)} = 1, \\
&k_{41}^{(2)} = -\frac{1}{\tilde{R}}\left(\frac{4}{r^3} + \frac{2\alpha}{r} + 2\beta r\right), \quad k_{42}^{(2)} = \frac{2\Omega}{\tilde{R}r}, \quad k_{43}^{(2)} = \frac{2\mathrm{i}}{\tilde{R}r^2}, \\
&k_{45}^{(2)} = -\frac{2\mathrm{i}}{\tilde{R}r^2}, \quad k_{46}^{(2)} = -\frac{2\mathrm{i}}{\tilde{R}r}, \quad k_{51}^{(2)} = -2\left(\frac{2\mathrm{i}}{r^2} - \Omega\right), \\
&k_{52}^{(2)} = \frac{4}{r^2} + 2\alpha + 2\beta r^2, \quad k_{54}^{(2)} = 2\mathrm{i}\tilde{R}, \quad k_{55}^{(2)} = \frac{1}{r}, \\
&k_{61}^{(2)} = -2\tilde{R}r, \quad k_{63}^{(2)} = \frac{3}{r^2} + 2\alpha + 2\beta r^2, \quad k_{66}^{(2)} = \frac{1}{r}.
\end{aligned}\right\} \tag{27}$$

The solution of the system (21), subject to the boundary conditions (22) and (23), was computed numerically, using a standard fourth-order Runge–Kutta scheme. Since the system (21) is singular at the centre of the pipe, it was necessary to start the numerical integration at a small distance away from the centre, using small-radius expansions of the regular homogeneous solutions and of a particular integral of (21). (For the complete details, which are quite involved owing to the presence of higher-order logarithmic terms in certain cases, the reader is referred to Demurger (1984).) It can be shown that the system of equations (21) has only three independent homogeneous solutions that satisfy the condition (22); the appropriate linear combination that satisfies the no-slip condition at the wall is determined by advancing numerically the three homogeneous solutions and a particular integral towards the wall, and imposing (23). Figures 3(a, b, c) show $U^{(2)}(r)$, $V^{(2)}(r)$, $W^{(2)}(r)$ for $\Omega = -26.96$, $\tilde{R} = 106.6$.

3.2. *The mean-flow distortion*

The amplitude $\boldsymbol{X}^{(0)}$ of the $O(\epsilon)$ mean-flow distortion induced by nonlinear effects is determined by solving the inhomogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}r}\boldsymbol{X}^{(0)} - \boldsymbol{K}^{(0)}\boldsymbol{X}^{(0)} = \boldsymbol{Q}^{(0)}, \tag{28}$$

$$U^{(0)} = V^{(0)} = W^{(0)} = P^{(0)} = 0 \quad (r = 0), \tag{29}$$

$$U^{(0)} = V^{(0)} = W^{(0)} = 0 \quad (r = 1); \tag{30}$$

the non-zero entries of $\boldsymbol{Q}^{(0)}$ are

$$Q_4^{(0)} = \frac{1}{r^2}\left(\frac{|U|^2}{r} + \frac{|V|^2}{r} - 2\mathrm{i}\frac{UV^*}{r} - 2\mathrm{i}UW^*\right) + \text{c.c.}, \tag{31}$$

$$Q_5^{(0)} = \frac{\tilde{R}}{r}(U^*F + \mathrm{i}W^*V) + \text{c.c.}, \tag{32}$$

$$Q_6^{(0)} = \frac{\tilde{R}}{r}\left(U^*G - \frac{U^*W}{r} + \mathrm{i}\frac{V^*W}{r}\right) + \text{c.c.}, \tag{33}$$

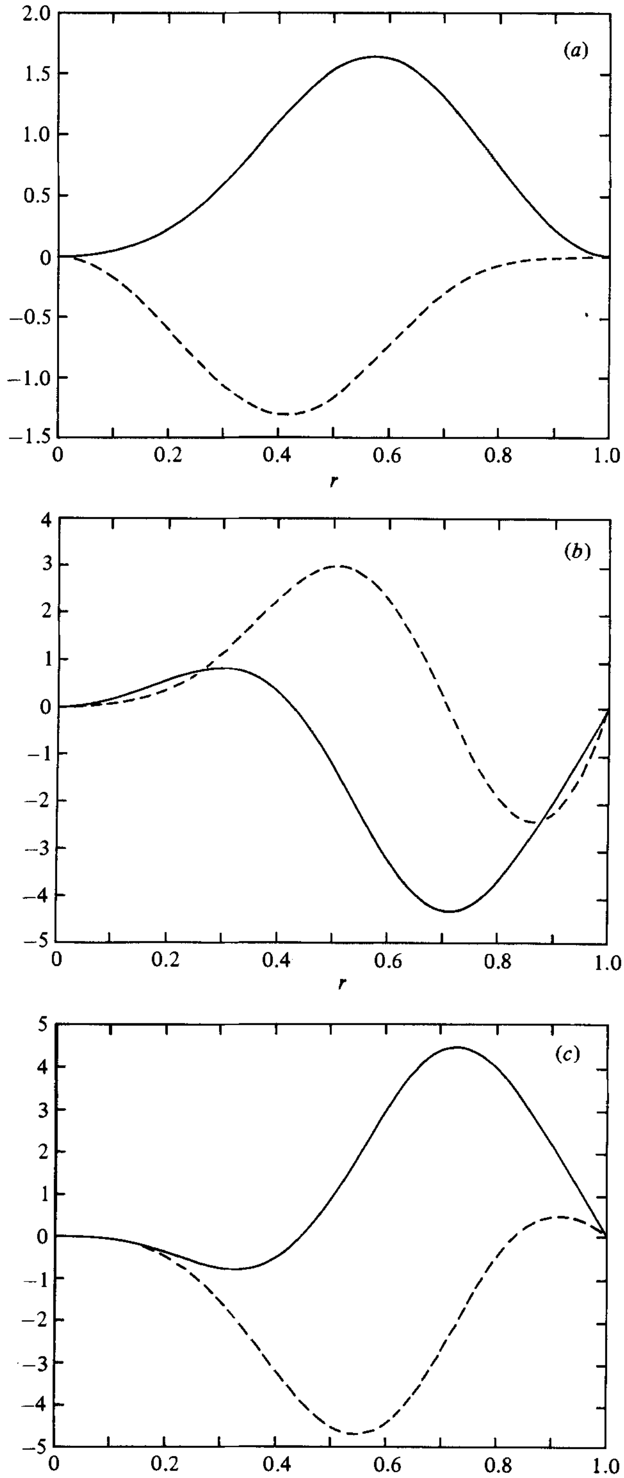

FIGURE 3. The modal amplitudes of the second harmonic for $\Omega = -26.96$, $\tilde{R} = 106.6$ (——, real part; – – –, imaginary part): (*a*) radial velocity component; (*b*) azimuthal velocity component; (*c*) axial velocity component.

while the non-zero elements of the matrix $\boldsymbol{K}^{(0)}$ are

$$\left.\begin{aligned} k_{25}^{(0)} = 1, \quad k_{36}^{(0)} = 1, \quad k_{42}^{(0)} = \frac{2\Omega}{\tilde{R}r}, \quad k_{51}^{(0)} = 2\Omega, \\ k_{55}^{(0)} = \frac{1}{r}, \quad k_{61}^{(0)} = -2\tilde{R}r, \quad k_{63}^{(0)} = -\frac{1}{r^2}, \quad k_{66}^{(0)} = \frac{1}{r}, \end{aligned}\right\} \tag{34}$$

The system of equations (28), subject to the boundary conditions (29) and (30), implies that

$$U^{(0)}(r) = 0; \tag{35}$$

the two independent homogeneous solutions of (28) that satisfy the conditions (29) at the centre are

$$(V^{(0)}, W^{(0)}, P^{(0)}) = (r^2, 0, \Omega r^2/\tilde{R}) \tag{36}$$

and

$$(V^{(0)}, W^{(0)}, P^{(0)}) = (0, r, 0). \tag{37}$$

The appropriate linear combination of (36) and (37) is specified by computing numerically a particular integral of (28) which satisfies (29) (using small-radius expansions near $r = 0$, as before) and imposing the no-slip condition (30) at the wall. The azimuthal and axial components of the induced mean flow $V^{(0)}$ and $W^{(0)}$ are shown in figures 4 (*a*, *b*) for $\Omega = -26.96$, $\tilde{R} = 106.6$. It is important to notice that the induced mean-flow components are typically $O(10)$ and not $O(1)$ as would normally be expected; as will be seen, this will have an important effect on the results of the non-linear analysis.

3.3. *The evolution equation for $A(T)$*

The evolution of the amplitude $A(T)$ is determined by considering the inhomogeneous problem for the $O(\epsilon^2)$ correction to the primary harmonic.† Returning to the expansion (20) and the governing equations (4)–(7), it is clear that the inhomogeneous terms of this problem are proportional to A^2A^* or $\mathrm{d}A/\mathrm{d}T$. Accordingly the relation

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \lambda A^2 A^* \tag{38}$$

is assumed, λ being the Landau constant, which will be specified by the requirement that the inhomogeneous problem for the $O(\epsilon^2)$ correction to the primary harmonic has a solution. To this end, the $O(\epsilon^2)$ correction to the primary harmonic is taken to be of the form $\epsilon^2 A^2 A^* \boldsymbol{Y} \mathrm{e}^{\mathrm{i}\phi} + \text{c.c.}$; (4)–(7) and the boundary conditions at the centre of the pipe and the wall then imply that the column vector $\boldsymbol{Y}(r)$ satisfies the inhomogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}r}\boldsymbol{Y} - \boldsymbol{K}\boldsymbol{Y} = \lambda \boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)}, \tag{39}$$

subject to the boundary conditions

$$Y_1 = Y_2 = Y_3 = Y_4 = 0 \quad (r = 0), \tag{40}$$

$$Y_1 = Y_2 = Y_3 = 0 \quad (r = 1). \tag{41}$$

† To $O(\epsilon^2)$ the nonlinear interactions also generate third harmonics, proportional to $\mathrm{e}^{3\mathrm{i}\phi}$, but these do not affect the evolution of $A(T)$, since they are not secular.

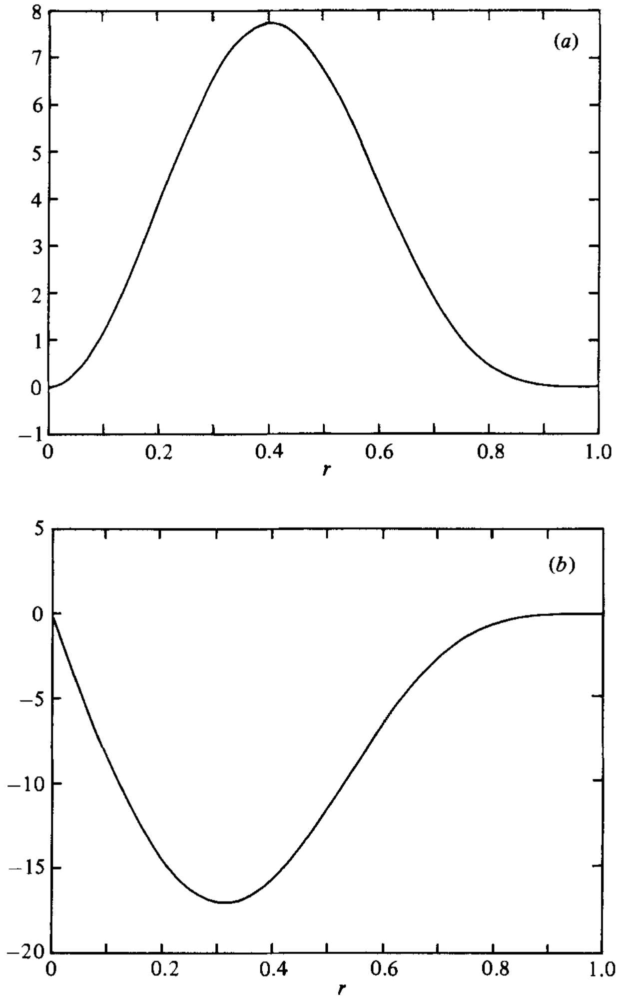

FIGURE 4. The induced mean flow for $\Omega = -26.96$, $\tilde{R} = 106.6$: (a) azimuthal velocity component; (b) axial velocity component.

The column vectors $\boldsymbol{Z}^{(1)}$ and $\boldsymbol{Z}^{(2)}$ have non-zero entries

$$Z_4^{(1)} = -\frac{U}{r}, \quad Z_5^{(1)} = \tilde{R}V, \quad Z_6^{(1)} = \tilde{R}W, \tag{42}$$

$$Z_4^{(2)} = \frac{1}{r^3}(3\mathrm{i}U^*V^{(2)} + 2U^*U^{(2)} - 3\mathrm{i}\,V^*U^{(2)} - \mathrm{i}\,V^{(0)}U + 2\,V^*V^{(2)} + 2\,V^{(0)}V)$$
$$+\frac{1}{r^2}(3\mathrm{i}U^*W^{(2)} - 3\mathrm{i}\,W^*U^{(2)} - \mathrm{i}UW^{(0)}), \tag{43}$$

$$Z_5^{(2)} = \mathrm{i}\frac{\tilde{R}}{r^2}(V^{(2)}V^* + V^{(0)}V)$$
$$+\frac{\tilde{R}}{r}(UF^{(0)} + U^*F^{(2)} + U^{(2)}F^* - \mathrm{i}\,V^*W^{(2)} + 2\mathrm{i}\,V^{(2)}W^* + \mathrm{i}\,W^{(0)}V), \tag{44}$$

$$Z_6^{(2)} = \frac{\tilde{R}}{r^2}(2\mathrm{i}\,V^* W^{(2)} - \mathrm{i}\,V^{(2)} W^* + \mathrm{i}\,V^{(0)} W - U W^{(0)} - U^* W^{(2)} - U^{(2)} W^*)$$

$$+\frac{\tilde{R}}{r}(UG^{(0)} + U^* G^{(2)} + U^{(2)} G^* + \mathrm{i}\,W^{(2)} W^* + \mathrm{i}\,W^{(0)} W). \tag{45}$$

Note that the matrix $\boldsymbol{K}$ in (39) is the one referred to in (15). Thus the homogeneous part of the problem (39)–(41) is exactly the same as the linear-stability eigenvalue problem (15), (18), (19). Since c is an eigenvalue, the inhomogeneous problem (39)–(41) does not have a solution, unless the Landau constant λ is chosen suitably.

To determine the appropriate solvability condition for the inhomogeneous problem (39)–(41) and thereby specify λ, the following procedure was adopted: the solution of the system (39), subject to the condition (40) at the centre of the pipe, may be written as

$$\boldsymbol{Y} = \sum_{i=1}^{3} C_i \boldsymbol{X}_i + \lambda \boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)}, \tag{46}$$

where $\boldsymbol{X}_i$ $(i = 1,2,3)$ are the three linearly independent regular solutions of the homogeneous system (15), C_i $(i = 1,2,3)$ are arbitrary constants, and $\boldsymbol{A}^{(j)}$ satisfy the inhomogeneous systems

$$\frac{\mathrm{d}\boldsymbol{A}^{(j)}}{\mathrm{d}r} - \boldsymbol{K}\boldsymbol{A}^{(j)} = \boldsymbol{Z}^{(j)} \quad (j = 1,2), \tag{47}$$

subject to the condition (40) at the centre. Thus, for $\boldsymbol{Y}$ to be the required solution of (39)–(41), it remains to ensure that the no-slip condition (41) is satisfied. This leads to the following matrix equation for the constants C_i:

$$\boldsymbol{X}\boldsymbol{C} = -\lambda \boldsymbol{a}^{(1)} - \boldsymbol{a}^{(2)}, \tag{48}$$

where $\boldsymbol{X}$ is the 3×3 matrix whose ith column consists of the first three entries of $\boldsymbol{X}_i$ evaluated at $r = 1$; $\boldsymbol{C}$ is the column vector with elements C_i, and $\boldsymbol{a}^{(1)}$, $\boldsymbol{a}^{(2)}$ are respectively the column vectors formed by the first three entries of the vectors $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$ evaluated at $r = 1$. However, the matrix $\boldsymbol{X}$ in (48) is singular, since c is an eigenvalue. Therefore the inhomogeneous system (48) does not have a solution unless the right-hand side is orthogonal to the solution $\boldsymbol{C}'$ of the corresponding transpose homogeneous system

$$\boldsymbol{C}'^{\mathrm{t}}(\lambda \boldsymbol{a}^{(1)} + \boldsymbol{a}^{(2)}) = 0, \tag{49}$$

where

$$\boldsymbol{X}^{\mathrm{t}}\boldsymbol{C}' = 0. \tag{50}$$

The solvability condition (49) specifies the Landau constant

$$\lambda = -\frac{\boldsymbol{C}'^{\mathrm{t}}\boldsymbol{a}^{(2)}}{\boldsymbol{C}'^{\mathrm{t}}\boldsymbol{a}^{(1)}}. \tag{51}$$

The Landau constant was calculated for eleven combinations of Ω and $\tilde{R}$ on the neutral-stability curve of the linear theory; the results are given in table 1. Most of the work involved in the calculation of λ is associated with the numerical solution of the inhomogeneous problem for $\boldsymbol{A}^{(2)}$ in (47); as already indicated, a small-radius expansion of the solution is required and tends to be a likely source of algebraic errors. As a check, the behaviour of the numerical solution for small r was investigated and was found to be consistent with the result of the small-radius expansion. In addition, the use of a refined integration step did not alter the results within the accuracy reported here.

Ω	$\tilde{R}$	$\lambda_r \times 10^{-4}$	$\lambda_i \times 10^{-4}$
−32	64.6	3.38	−0.27
−31	68.2	3.43	−0.29
−30	72.5	3.50	−0.32
−29	77.9	3.61	−0.37
−28	85.4	3.80	−0.46
−26.96	106.6	4.46	−0.88
−28	131.4	5.30	−1.68
−29	142.0	5.65	−2.10
−30	150.6	5.90	−2.46
−31	157.9	6.10	−2.79
−32	164.5	6.27	−3.10

TABLE 1. Eleven values of the Landau constant $\lambda = \lambda_r + i\lambda_i$ on the linear neutral-stability curve

4. Discussion

The main outcome of the present analysis are the values of the Landau constant listed in table 1. From these values, several conclusions can be drawn concerning the finite-amplitude stability of rotating pipe flow at high Reynolds number: first, the flow is subcritically unstable, since the real part of λ is positive ($\lambda_r > 0$); finite-amplitude perturbations are nonlinearly unstable outside (and close to) the neutral-stability curve of the linear theory (see figure 1). More precisely, from the evolution equation (38) it follows that the magnitude of the complex amplitude A is governed by

$$\frac{d|A|^2}{dt} = 2c_i|A|^2 + 2\epsilon^2\lambda_r|A|^4. \tag{52}$$

The equation (52) includes the effect of the exponential growth or decay predicted by the linear theory away from the linear neutral-stability curve and is a straightforward generalization of (38). (Formally, for (52) to be valid, c_i has to be $O(\epsilon^2)$.) Thus, for $c_i < 0$, and since $\lambda_r > 0$, (52) implies that there exists a threshold amplitude above which nonlinear (subcritical) instability occurs. Accordingly, the amount of rotation required to destabilize pipe flow is even smaller than that predicted by the linear theory; the suggestion of Mackrodt (1976) regarding the role of a small amount of extraneous rotation in the transition to turbulence of pipe flow appears to gain further support by the nonlinear theory.

There is a further aspect of the results of the nonlinear analysis, however, which deserves special attention: the surprisingly large magnitudes of the Landau constant. For example, in plane Poiseuille flow, at the critical Reynolds number for linear instability ($R = 5772$), λ_r is three orders of magnitude smaller than the present values (Reynolds & Potter 1967). One could argue that this difference is due to the assumption of a large Reynolds number; however, the numerical calculations of Cotton & Salwen (1981) indicate that the limit $\mu \to 0$, $R \to \infty$, $\tilde{R}$ finite, considered here, has been reached at Reynolds numbers as low as $R = 1000$. Thus one is led to conclude that, in the present problem, nonlinearity is particularly strong. The cause of the large magnitudes of λ is the relatively large mean-flow distortion induced by the Reynolds stress (see figures 4a,b).

The large magnitudes of λ suggest that the validity of the Stuart–Watson expansion (20) is restricted to very small values of ϵ; the evolution of unstable perturbations is very soon governed by fully nonlinear processes, which cannot be described by weakly nonlinear theories. In this sense, the information provided by a linear stability analysis is very limited.

The possibility of large Landau constants has been considered by Davis & Rosenblat (1977), in certain model problems. They find that, as the magnitude of the Landau constant increases, the region of validity of the weakly nonlinear expansion shrinks until, finally, the linear theory is completely invalidated in the limit of infinite Landau constant, and the evolution of perturbations to leading order is determined by a fully nonlinear balance. It is interesting to note that, in the case of rotating-pipe flow, the magnitude of the Landau constant increases as $\Omega/\tilde{R}$ decreases on the neutral-stability curve (see table 1). This suggests that the region of validity of the expansion (20) shrinks, as the ratio of the azimuthal to the axial basic-flow velocity component decreases; that is, it appears that nonlinearity becomes increasingly more important as rotating-pipe flow tends to be 'closer' to non-rotating pipe flow. This interesting possibility could be investigated by numerical experiments with the full nonlinear problem.

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